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A variational approach of nonlinear dissipative pulse propagation

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Abstract. A variational technique to deal with nonlinear dissipative pulse propagation is established. By means of a generalization of the Kantorovitch method, suitable for non-conservative systems, we are able to cope with an extended nonlinear Schrödinger equation (NLSE) which describes pulse propagation under the influence of nonlinear loss and/or gain, in particular, in the presence of two-photon absorption (TPA). Based on the characteristics of the exact solution of the NLSE in the absence of TPA, we investigate the effects of frequency dispersion of the nonlinear susceptibility associated to the two-photon resonance, obtaining the necessary conditions for a solitary wave solution, even in the presence of a self-steepening term.

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Variational approaches have gained a new thrust in recent years in many branches of science, including quite distinct fields such as correlations in charged polymers [1], sandpile dynamics [2] and excitations of a Bose-Einstein condensate [3]. The variational analysis provides us with a formalism capable to establish fundamental equations that describe qualitatively and quantitatively the dynamical behavior of a specific system. On the other hand, with the laser's advent, nonlinear optics has become a particularly interesting field for its theoretical context as well as its practical consequences to technology, in particular, to nonlinear optical fiber and waveguide systems. Here too, the variational method has been widely applied to obtain approximated solutions for problems concerning pulse and/or beam propagation within the framework of the nonlinear Schrödinger equation (NLSE) which applies to problems involving one or more *transverse dimensions* besides the propagation dimension.

However the application of the variational method to non-conservative optical systems has been very limited [4–6], because it has been assumed that non-conservative aspects of pulse and/or beam propagation either could not be treated or it was very complicated using this type of analysis. Nevertheless it would be quite important to have a valid mathematical framework that could provide approximated solutions for problems in propagation that involve nonlinear loss and/or gain processes. The need for alternative solutions lies on the fact that rigorous ones can usually be found only for oversimplified models of physical situations.

An example of such an actual problem is nonlinear propagation in the vicinity of a two-photon absorption. It is well known that third order nonlinearities may be enhanced near such a resonance [7,8]. In particular, lately there has been considerable interest in the two-photon resonant enhanced nonlinearity near the half band gap of semiconductors [9] for all-optical switching applications, where TPA can be a limiting factor [10,11]. Pulse propagation studies in this situation [12,13] indicate asymmetric frequency spectra and self-steepening of the transmitted pulse as a consequence of frequency dispersion of the nonlinear refractive index $n_2(\omega)$ and also of the TPA coefficient $\kappa(\omega)$. Therefore, to describe properly pulse propagation in the vicinity of a TPA , one has to deal with an extended NLSE that includes terms that are originated from the frequency dispersion of the nonlinear susceptibility in addition to the nonlinear absorption. There is no known exact solution for such equation so that variational solutions would be quite useful here.

In this paper we firstly establish a mathematical framework based on variational principles to find approximated solutions for an extended nonlinear Schrödinger equation that includes nonlinear non-conservative processes which might occur during propagation. To this end, we set up suitable Euler-Lagrange equations and solve them by means of the Kantorovitch ansatz for a Lagrangian function with two independent variables.

After that, to demonstrate the power of our method, we apply it to study the behavior of pulses propagating

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near a two-photon resonance. First by using the exact solution in the absence of TPA to find analytical expressions for the dynamics of the soliton's parameters. Next we include a self-steepening term, which takes into account the frequency dispersion of the nonlinear susceptibility and find a suggestive picture of the interplay among the various effects. In particular we show that under certain particular conditions it is still possible to find solitary wave solutions.

Let us begin by writing the total Lagrangian of the system as the sum of two terms, a conservative one L_{C} and a non-conservative one, L_{NC}:

$$L(u, u^*, \xi, \tau, u_\xi, u_\tau, u_\xi^*, u_\tau^*) = L_{\mathrm{C}} + L_{\mathrm{NC}} \ , \qquad (1)$$

where $u(\xi, \tau)$ represents the amplitude envelope of the optical field, which is a slowly varying function of the time τ and propagation distance ξ. The fundamental problem here is to find out the extreme functions $u(\xi, \tau)$ that renders the Langrangian integral stationary. This may be expressed as a Hamilton's principle [14],

$$\delta \left[\int \int L \mathrm{d}\xi \mathrm{d}\tau \right] = \delta \left[\int \int (L_{\mathrm{C}} + L_{\mathrm{NC}}) \mathrm{d}\xi \mathrm{d}\tau \right] = 0 \ , \quad (2)$$

so the Euler-Lagrange equations describing the dynamics of the system are given by,

$$\frac{\delta L}{\delta u_i} = \frac{\partial}{\partial \xi} \frac{\partial L_{\mathrm{C}}}{\partial (\frac{\partial u_i}{\partial \xi})} + \frac{\partial}{\partial \tau} \frac{\partial L_{\mathrm{C}}}{\partial (\frac{\partial u_i}{\partial \tau})} - \frac{\partial L_{\mathrm{C}}}{\partial u_i} = Q_i \ . \qquad (3)$$

Here, the index i runs through $1 - 2$ with $u_1 = u$ and $u_2 = u^*$. The function Q_i takes into account all non-conservative processes described by the corresponding terms of the total Lagrangian and is given by,

$$Q_i = \frac{\partial L_{\mathrm{NC}}}{\partial u_i} - \frac{\partial}{\partial \xi} \frac{\partial L_{\mathrm{NC}}}{\partial (\frac{\partial u_i}{\partial \xi})} - \frac{\partial}{\partial \tau} \frac{\partial L_{\mathrm{NC}}}{\partial \frac{\partial u_i}{\partial \tau}} \cdot \qquad (4)$$

To find approximated solutions for the Euler-Lagrange equations for non-conservative systems, we can use a generalization of the Rayleigh-Ritz method known as Kantorovitch method. The method assumes that the extremum of the variational integral of the Lagrangian function may be expressed as,

$$u(\xi, \tau) = f(b_1(\xi), b_2(\xi), ..., b_N(\xi), \tau) \ , \qquad (5)$$

where f is a guessed function based on a previous knowledge of the system's behavior and b's are N unknown parameter functions to be determined. The substitution of the constants parameters of the Rayleigh-Ritz method by functions is the base of Kantorovitch method. Thus the Lagrangian depend on the independent variables ξ and τ and also on the dependent variables $u(\xi, \tau)$ and $u^*(\xi, \tau)$, which in turn depends on the dependent variables b_j through the known function f. Just as before, our problem is to find the extreme of the variational integral,

$$I = \int \int L(\xi, \tau, f(b_1(\xi), b_2(\xi), ..., b_N(\xi), \tau), \qquad (6)$$
$$f^*(b_1(\xi), b_2(\xi), ..., b_N(\xi), \tau), u_\xi, u_\tau, u_\xi^*, u_\tau^*) \mathrm{d}\xi \mathrm{d}\tau.$$

Since the dependence of the integrand with respect to the variable τ is known, the integration in τ may be performed. However, for the moment we leave it explicitly and note that the only independent functions are b_j. Performing the variation supposing that the only independent variable is ξ and using the implicit function theorem we find,

$$\sum_{i=1}^{n} \int \int \left(\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\partial L_{\mathrm{C}}}{\partial f_\xi} - \frac{\partial L_{\mathrm{C}}}{\partial f} + Q_u(f) \right) \frac{\partial f}{\partial b_i} \delta b_i \mathrm{d}\xi \mathrm{d}\tau = 0 \ . \qquad (7)$$

Now, we may restate the variational problem , in terms of the variable ξ and of the parameter functions b's, as,

$$\sum_{i=1}^{n} \int \int \left(\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\partial L_{\mathrm{KC}}}{\partial (b_i)_\xi} - \frac{\partial L_{\mathrm{KC}}}{\partial b_i} + Q_{\mathrm{K}} \right) \delta b_i \mathrm{d}\xi \mathrm{d}\tau = 0 \quad (8)$$

where the subindex K is used to make the distinction of the fuctional form before and after substitution of b's.

Identifying terms, conservative and non-conservative, between the last two equations and defining $< L_{\mathrm{KC}} >= \int L_{\mathrm{KC}} \mathrm{d}\tau$ we finally obtain the Euler-Lagrange equations for the functions b_j,

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\partial < L_{\mathrm{KC}} >}{\partial (b_i)_\xi} - \frac{\partial < L_{\mathrm{KC}} >}{\partial b_i} = \int Q_{\mathrm{K}} \frac{\partial u}{\partial b_i} \mathrm{d}\tau \ . \qquad (9)$$

The set of equations (9) need to be solved to get an approximated solution of the extremum that minimizes the integral given by equation (2). In the case that the Lagrangian function describes a conservative physical system, the right hand side of equation (9) is identically zero, yielding to the Euler-Lagrange equations for conservative systems.

Now to illustrate and check the effectiveness of the formalism developed, let us consider the following dissipative NLSE describing the propagation in the presence of TPA,

$$i \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = -i\kappa |u|^2 u \ , \qquad (10)$$

where κ represents the TPA coefficient, that is $\kappa \propto \mathrm{Im}\chi^3$, u, ξ and τ are normalized variables representing electric field amplitude, longitudinal coordinate and orthogonal coordinate (space or time). For the sake of simplicity we are using just $1+1$ dimensions. The Lagrangian corresponding to the conservative problem ($\kappa = 0$) is given by:

$$L_{\mathrm{C}} = \frac{i}{2} \left(u \frac{\partial u^*}{\partial \xi} - u^* \frac{\partial u}{\partial \xi} \right) + \frac{1}{2} \left| \frac{\partial u}{\partial \tau} \right|^2 - \frac{1}{2} |u^4| \ . \qquad (11)$$

Here, the non-conservative process is described by:

$$Q = -i\kappa |u|^2 u. \qquad (12)$$

The Lagrangian and the non-conservative functions have u and its complex conjugate as dependent variables. Notice that although u and u^* are related through the complex conjugation operator, they are linearly independent.

Furthermore, the Euler-Lagrange equations for these variables are related through:

$$\frac{\delta L}{\delta u^*} = \left(\frac{\delta L}{\delta u}\right)^* = Q_{\mathrm{K}} \ . \tag{13}$$

For this kind of situation, the Kantorovitch method as given in equation (9) can be modified to read,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\partial < L_{\mathrm{KC}} >}{\partial (b_i)_\xi} - \frac{\partial < L_{\mathrm{KC}} >}{\partial b_i} = 2\mathrm{Re}\int Q_{\mathrm{K}}\frac{\partial u^*}{\partial b_i}\mathrm{d}\tau \ . \tag{14}$$

Next, an ansatz is proposed for $u(\xi,\tau)$ based on the features of the solution in the absence of non-conservative processes. When the TPA term is null, the NLSE has an exact solution with the following functional form:

$$u(\xi,\tau) = A(\xi)\mathrm{sech}\left(\frac{\tau}{\omega(\xi)}\right)\exp(i\phi(\xi)) \ . \tag{15}$$

Therefore, we propose a solution of this form and expect it to evolve adiabatically when perturbed by a TPA process.Here it should be noted that this ansatz yields only to adiabatic soliton solutions, excluding any dynamical oscillations. After inserting equation (15) into the Lagrangian function, integrating with respect to τ performing all the variations with respect to the function parameters A, ω and ϕ and applying equation (14) we find that the dynamic equations are,

$$2\frac{\partial\phi}{\partial\xi} - \frac{1}{3\omega^2} - \frac{2}{3}A^2 = 0 \ , \tag{16}$$

$$2\frac{\partial\phi}{\partial\xi} + \frac{1}{3\omega^2} - \frac{4}{3}A^2 = 0 \ , \tag{17}$$

$$\frac{\partial}{\partial\xi}(2\omega A^2) = -\frac{8}{3}\kappa\omega A^4 \ . \tag{18}$$

It should be noted here that this last equation expresses the fact that the pulse power dissipation is proportional to A^4, corresponding to the absorption of two photons. This set of equations is easily solved to determine the evolution of the soliton's parameters and produce exactly the same solution obtained in reference [15] *via* a different route. Having shown the effectiveness of the present scheme, we now turn to a practical problem with no known exact solution.

There has been evidences that pulse propagation near a two-photon resonance produces asymmetric frequency spectra together with self-frequency shift of the transmitted pulse. Therefore, we must incorporate into the above idealized model, the frequency dispersion of the real part of the third order susceptibility. To this end we use the wave equation derived in reference [13],

$$i\frac{\partial u}{\partial\xi} + \frac{1}{2}\frac{\partial^2 u}{\partial\tau^2} + |u|^2 u - i\alpha|u|^2\frac{\partial u}{\partial\tau} = -i\kappa|u|^2 u \ , \tag{19}$$

where $\alpha \propto \frac{\partial \mathrm{Re}(\chi)^3}{\partial\omega}$ is the self-steepening parameter. The Lagrangian corresponding to the conservative problem is now given by:

$$L_{\mathrm{C}} = \frac{i}{2}\left(u\frac{\partial u^*}{\partial\xi} - u^*\frac{\partial u}{\partial\xi}\right) + \frac{1}{2}\left|\frac{\partial u}{\partial\tau}\right|^2 - \frac{1}{2}|u|^4 - \frac{i}{2}\alpha|u|^2 u\frac{\partial u^*}{\partial\tau} \tag{20}$$

and the non-conservative process is still described by equation (12). We propose an ansatz capable to deal with the self-steepening problem, that is,

$$u(\xi,\tau) = \rho\exp[-i\phi(\xi,\tau)] \ , \tag{21}$$

$$\rho(\xi,\tau) = A(\xi)\mathrm{sech}\left[\frac{\tau}{\omega(\xi)}\right] \ , \tag{22}$$

$$\frac{\partial\phi}{\partial\tau} = a_0 + a_2\rho^2 \ , \tag{23}$$

where a_0 and a_2 are constants. By proceeding as before, we end up with a set of Euler-Lagrange equations, whose solutions we may write as:

$$A(\xi) = \left\{\Gamma(0) + \frac{8}{3}K\xi - \frac{8\eta}{5\mu}\ln\left[\frac{A^2(\xi)}{\frac{8}{5}\eta A^2(\xi) - \mu}\right]\right\}^{-\frac{1}{2}} \ , \tag{24}$$

$$\omega(\xi) = \frac{\omega(0)}{A(\xi)\sqrt{\mu - \frac{8}{5}\eta A^2(\xi)}} \ , \tag{25}$$

$$\phi(\xi) = \phi(0) + \frac{a_0^2}{2}\xi - \frac{\mu}{2}\int A^2(\xi)\mathrm{d}\xi - \frac{8\eta}{15}\int A^4(\xi)\mathrm{d}\xi \tag{26}$$

where

$$\mu = 1 + a_0(\alpha - 2a_2) \ , \tag{27}$$

$$\eta = a_2(\alpha - a_2) \ . \tag{28}$$

where $\Gamma(0)$ is a constant related to $A(0)$. Notice that when $\alpha - a_2 = 0$, we find a solitary wave solution , recovering the same solution class obtained in the absence of self-steepening and asymmetric self-phase modulation, which is a completely new result concerning this problem [13]. Physically it can be understood that the initial chirp imposed on the pulse through the parameter a_2 determines that during propagation the pulse can balance the self-steepening parameter α associated with the material. It is also possible to show that there are combinations of a_2 and α that produces enhancement or attenuation of theself-steepening.

In conclusion we have established a new variational formalism suited to deal with nonlinear dissipative Schrödinger equations and showed the existence of solitary wave solutions in TPA medium even when the pulse wavelength was so near the band gap that self-steepening played an important role. Our method has an advantage over those found in the literature for non-conservative systems [4–6], because there is no need to know *a priori* the decaying/growing functional form of the solution and also we can easily deal with nonlinear loss or gain.

The resulting formalism may be generalized to deal with propagation problems that involve higher dimensions and quadratic nonlinear media [16] as well as other types of non-conservative nonlinearities such as found in the Ginzburg-Landau equation which describes pulse propagation in Er^{3+}-doped fibers. This work was partially supported by the Brazilian agencies: FINEP, CNPq and CAPES.

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